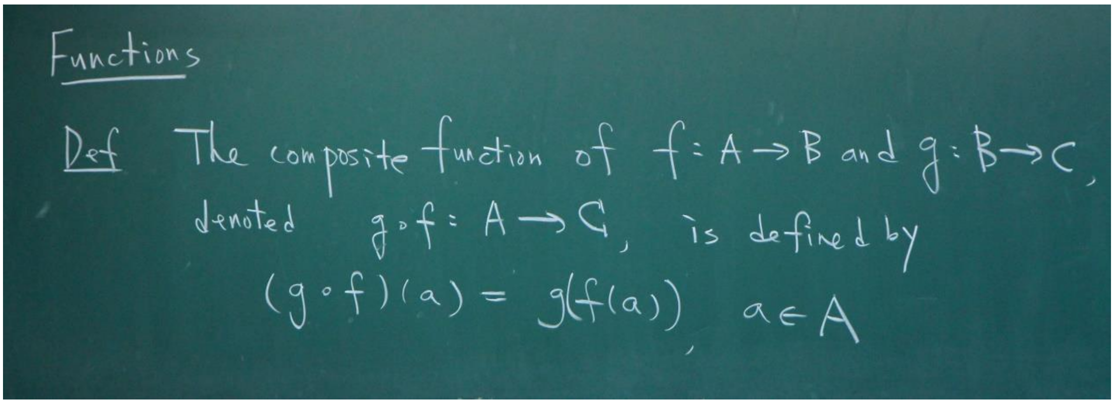
【10920 趙啟超教授離散數學 / 第 4 堂版書】



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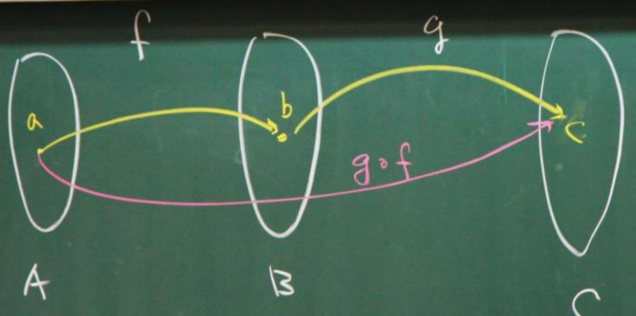
$$(g \circ f)(a) = g(f(a)), \quad a \in A$$



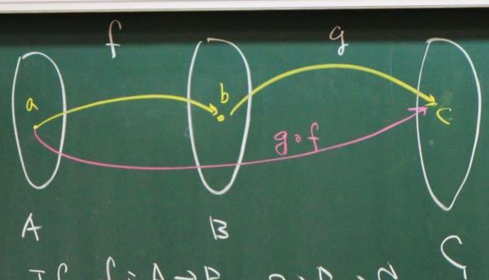
<u>Remark</u> If $f : A \to B$, $g : B \to C$, and $h : C \to D$,

$$(h \circ g) \circ f = h \circ (g \circ f)$$

<u>Functions</u>

<u>Def</u> The composite function of $f : A \to B$ and $g : B \to C$,
denoted $g \circ f : A \to C$, is defined by

$$(g \circ f)(a) = g(f(a)), \quad a \in A$$



<u>Remark</u> If $f : A \to B$, $g : B \to C$, and $h : C \to D$,

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Property $f : A \to B$ and $g : B \to C$

① If f and g are injective, then so is $g \circ f$.

② If f and g are surjective, then so is $g \circ f$.

③ If f and g are bijective, then so is $g \circ f$.

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③ If f and g are bijective, then so is $g \circ f$.

Proof ① Suppose $(g \circ f)(a_1) = (g \circ f)(a_2)$.

$$\Rightarrow \quad g(f(a_1)) = g(f(a_2))$$

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$\therefore$ $g \circ f$ is injective.

② If g is surjective, then for all $c \in C$, there exists some $b \in B$ with $g(b) = c$,

and if f is surjective, there exists some $a \in A$ with $f(a) = b$.

Thus $c = g(b) = g(f(a)) = (g \circ f)(a)$.

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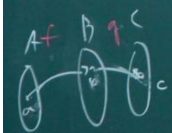

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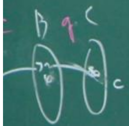

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Therefore, $g \circ f$ is surjective.

③ The proof follows directly from ① and ②. ∎

Def A function $f: A \to B$ is said to be invertible if there is a function $g: B \to A$, such that

$(g \circ f)(a) = a$, $a \in A$, $(f \circ g)(b) = b$, $b \in B$

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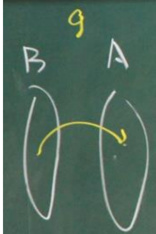
Remark g is called an inverse of f.

Property If f has an inverse, then it is unique.

Proof Suppose g and g' both satisfy the condition for an inverse of $f: A \to B$, i.e.,

$$(g\circ f)(a) = a, \quad (f\circ g)(b) = b$$



$$(g'\circ f)(a) = a, \quad (f\circ g')(b) = b.$$

For all $b \in B$,

$$\begin{aligned} & g(b) \\ = & (g'\circ f)(g(b)) \\ = & g'(f(g(b))) = g'((f\circ g)(b)) \\ = & g'(b) \quad \therefore g = g'. \end{aligned}$$

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Notation: Inverse of f: f^{-1}

Note that $(f^{-1})^{-1} = f$.

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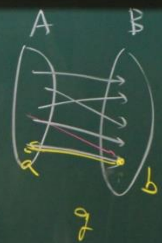
<u>Proof</u> "$\Leftarrow$" Suppose $f: A \to B$ is bijective.

For all $b \in B$, there exists a unique $a \in A$ such that $f(a) = b$.

The rule $g(b) = a$ defines a function from B to A.



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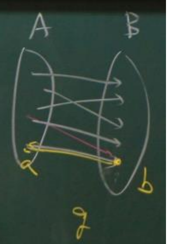

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The rule $g(b) = a$ defines a function from B to A.

Then $(g \circ f)(a) = g(f(a)) = g(b) = a$

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Hence g is an inverse of f and f is invertible.

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For all $b \in B$, we know that $f(f^{-1}(b)) = b$.

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So letting $a = f^{-1}(b) \in A$ yields $f(a) = b$.

Thus f is surjective.

Now suppose $f(a_1) = f(a_2)$, for $a_1, a_2 \in A$

$\Rightarrow$ $f^{-1}(f(a_1)) = f^{-1}(f(a_2))$

$\Rightarrow$ $a_1 = a_2$

Hence f is injective. Therefore, f is bijective.

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